§4.2 Conservation of energy

Let us summarize what we have discussed so far:

$$K_2 - K_1 = \int_{x_1}^{x_2} F(x)\,dx = G(x_2) - G(x_1) = G_2 - G_1$$

rearranging gives

$$K_2 - G_2 = K_1 - G_1 \qquad (*)$$

Introduce the function

$$U(x) = -G(x) \qquad F(x) = -\frac{dU}{dx}$$

$\longrightarrow$ $(*)$ becomes

$$\underbrace{K_2 + U_2}_{=:E_2} = \underbrace{K_1 + U_1}_{=:E_1}$$

Have arrived at

<u>Theorem 2</u> (law of conservation of energy):

The quantity $E = K + U = \frac{1}{2} m v^2 + U(x)$ does not change with time, i.e is "conserved" throughout time

Remark:

$E = K + U$ is called "total mechanical energy" and U is called "potential energy".

Example 1:

i) Suppose we drop a rock from a certain height h $\rightarrow$ total mechanical energy $E = \frac{1}{2} m v^2 + U(y)$ has to be conserved throughout the fall:

$$F = -mg \implies U(y) = mgy$$

$$\text{as } -\frac{dU}{dy} = -mg = F$$

$\rightarrow$ energy conservation law becomes:

$$E_2 = \frac{1}{2} m v_2^2 + mgy_2 = \frac{1}{2} m v_1^2 + mgy_1 = E_1$$

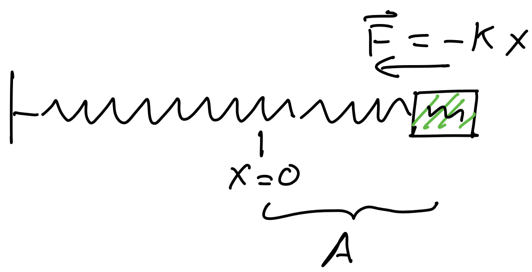
ii) In the mass and spring system the corresponding relations are:

$$U(x) = \frac{1}{2} k x^2 \quad \text{as} \quad -\frac{dU}{dx} = -kx = F(x)$$

giving $E_2 = \frac{1}{2} m v_2^2 + \frac{1}{2} k x_2^2 = \frac{1}{2} m v_1^2 + \frac{1}{2} k x_1^2 = E_1$

Let us apply this to a concrete problem: pull the spring by an amount A

Question: What is the velocity when the mass comes back to $x = 0$?



"harmonic motion"

Setting $x_1 = A$ and $v_1 = 0$ in the energ law gives:

$$\frac{1}{2} m v^2 + \frac{1}{2} k x^2 = 0 + \frac{1}{2} k A^2 \quad (**)$$

Then, at $x=0$ we get

$$\frac{1}{2} m v^2 + \frac{1}{2} k \cdot 0^2 = 0 + \frac{1}{2} k A^2$$

$$\Leftrightarrow \quad v^2 = \frac{kA^2}{m}$$

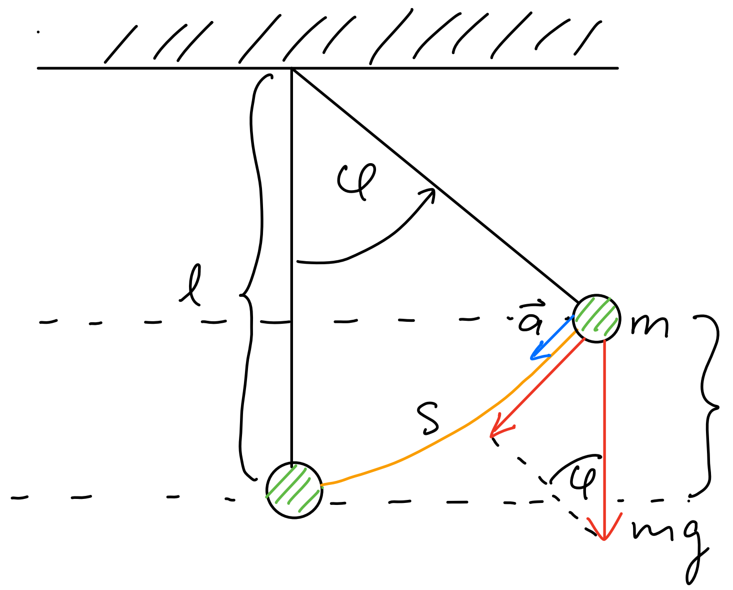
$$v = \pm \sqrt{\frac{kA^2}{m}} = \pm A \sqrt{\frac{k}{m}} = \pm \omega A$$

$\rightarrow$ two answers as mass can be going in either direction

In general: $(**) \Leftrightarrow v = \pm \sqrt{\frac{k}{m}} \sqrt{A^2 - x^2}$

iii) Pendulum:

Imagine a mass m hanging from the ceiling along a thread of length ℓ



length s is given by $s = \ell\varphi$

$\rightarrow \quad h = \ell(1-\cos\varphi)$

For <u>small</u> φ, we can Taylor expand:

$$\cos\varphi \approx 1 - \frac{1}{2}\varphi^2$$

$$\Rightarrow \quad h \approx \frac{\ell\varphi^2}{2} = \frac{s^2}{2\ell}$$

$\rightarrow$ potential energy becomes:

$$E_{pot} = mgh \approx mg\frac{s^2}{2\ell} = \frac{1}{2}Ds^2, \quad D = \frac{mg}{\ell}$$

$\rightarrow$ motion is harmonic!

§4.3 Conservation of energy in $d>1$

Let us summarize the situation $d=1$:

$$\frac{dK}{dt} = mv\frac{dv}{dt} = mva = Fv = F\frac{dx}{dt}, \quad K = \frac{1}{2}mv^2$$

$$dK = F\,dx \quad \text{(upon cancelling } dt\text{)}$$

$$K_2 - K_1 = \int_{x_1}^{x_2} F(x)\,dx$$

$$= U(x_1) - U(x_2)$$

$$K_2 + U_2 = K_1 + U_1$$

Now let us look at $d=2$:

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(v_x^2 + v_y^2)$$

$$\Leftrightarrow \frac{dK}{dt} = m\left(v_x\frac{dv_x}{dt} + v_y\frac{dv_y}{dt}\right)$$

$$= F_x v_x + F_y v_y = F_x\frac{dx}{dt} + F_y\frac{dy}{dt}$$

$$dK = F_x\,dx + F_y\,dy$$

$d \geq 2$:

Denote the position of a point-like

mass by $\vec{r}(t)$. Then $\vec{v}(t) = \dot{\vec{r}}(t)$ and

$$\underbrace{\frac{d}{dt}\left(\frac{1}{2}m\dot{\vec{r}}^{2}\right)}_{\substack{\text{change of}\\ \text{Kinetic energy}\\ \text{per unit time}}} = \frac{1}{2}m\frac{d}{dt}(\dot{\vec{r}}\cdot\dot{\vec{r}})$$

$$= \frac{1}{2}m\left(\ddot{\vec{r}}\cdot\dot{\vec{r}} + \dot{\vec{r}}\cdot\ddot{\vec{r}}\right)$$

$$= m\,\ddot{\vec{r}}\cdot\dot{\vec{r}} = \underbrace{\vec{F}\cdot\dot{\vec{r}}}_{\substack{\text{work done}\\ \text{per unit time}}}$$

We define the infinitesimal work done as:

$$dW = \vec{F}\cdot d\vec{r} = \sum_{i=1}^{d} F_i\, dr_i = dK \quad (*)$$

$$\left(\text{in } d=2 \;\; \underset{=}{} \; F_x\,dx + F_y\,dy\right)$$

and power as

$$P = \frac{dK}{dt} = \vec{F}\cdot\dot{\vec{r}}$$

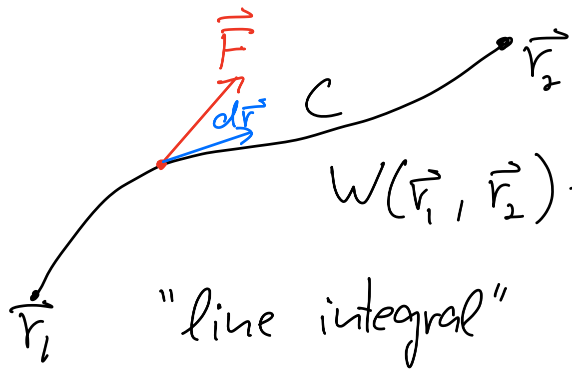
For vectors we have the formula

$$\vec{A}\cdot\vec{B} = AB\cos\theta \quad , \quad A = |\vec{A}|\,,\; B = |\vec{B}|$$

where θ is the angle suspended between the vectors $\vec{A}$ and $\vec{B}$.

→ for a constant force: $W = F\Delta r \cos(\vec{F}, \Delta\vec{r})$
$= \vec{F}\cdot\Delta r$

Equation (*) gives for the work done between two positions $\vec{r}_1$ and $\vec{r}_2$ along a curve $C(\vec{r}_1, \vec{r}_2)$:



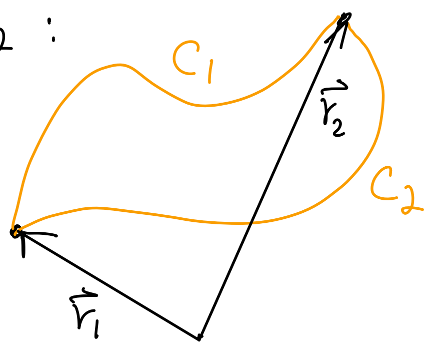
$$W(\vec{r}_1, \vec{r}_2) = \int_{C(\vec{r}_1, \vec{r}_2)} \vec{F}(\vec{r}) \cdot d\vec{r}$$

"line integral"

$$= \lim_{\Delta\vec{r} \to 0} \sum \underbrace{\Delta W}_{\approx \vec{F} \cdot \Delta\vec{r}}$$

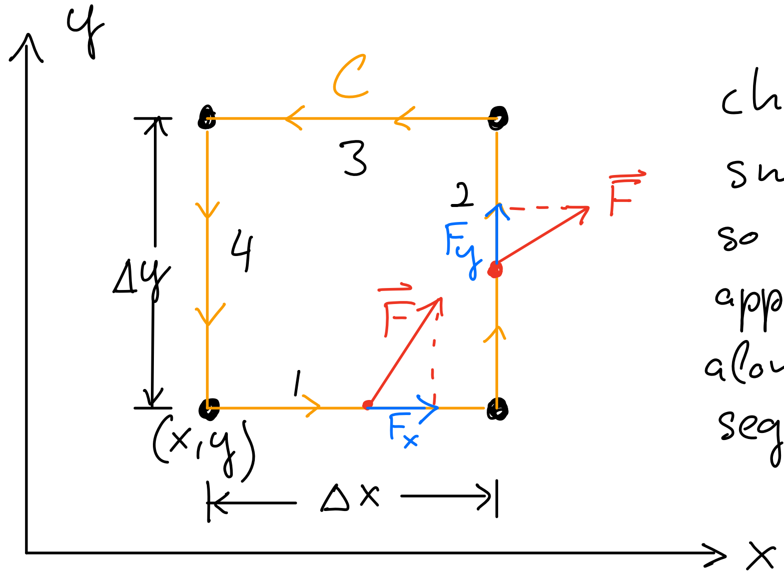
§ 4.4 Conservative and non-conservative forces

Suppose I go from $\vec{r}_1$ to $\vec{r}_2$ along a path C_1 and someone else goes along C_2 :



Is the work done the same, i.e does $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ hold ?

To get a clue, let us consider a specific example, namely a small square:



choose $\Delta x, \Delta y$ small enough so that $\vec{F}$ is approx. const. along each segment

→ Want to compute $\oint_C \vec{F} \cdot d\vec{r}$ along the direction of arrows

Contributions along different sides:

(1) the tangential component is $F_x(1)$

→ $\vec{F} \cdot \Delta\vec{r} = F_x(1)\,\Delta x$

(2) $\vec{F} \cdot \Delta\vec{r} = F_y(2)\,\Delta y$

(3) $\vec{F} \cdot \Delta\vec{r} = -F_x(3)\,\Delta x$

(4) $\vec{F} \cdot \Delta\vec{r} = -F_y(4)\,\Delta y$

} minus sign due to reversed arrows

Together:

$$\oint \vec{F}\cdot d\vec{r} = F_x(1)\Delta x + F_y(2)\Delta y - F_x(3)\Delta x - F_y(4)\Delta y$$

rearrange:

$$(1) + (3) = [F_x(1) - F_x(3)]\Delta x$$

use $F_x(3) = F_x(1) + \frac{\partial F_x}{\partial y}\Delta y$

$$= -\frac{\partial F_x}{\partial y}\Delta x \Delta y$$

similarly

$$(2) + (4) = F_y(2)\Delta y - F_y(4)\Delta y = \frac{\partial F_y}{\partial x}\Delta x \Delta y$$

$$\rightarrow \oint \vec{F}\cdot d\vec{r} = \underbrace{\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)}_{(\vec{\nabla}\times\vec{F})_z}\underbrace{\Delta x\Delta y}_{=\Delta a}$$

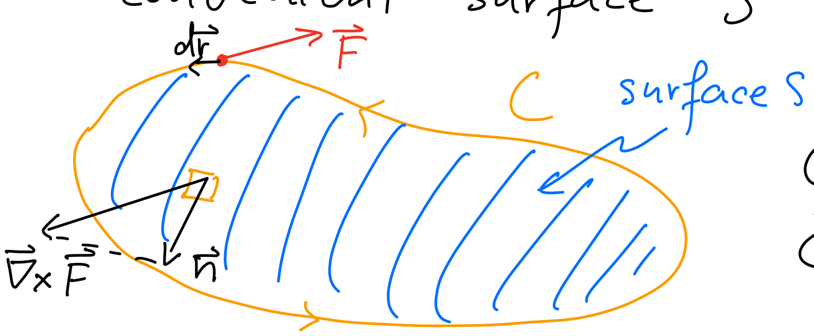
area

curl

normal vector

$$= (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,\Delta a$$

→ Now fill a given loop C with any convenient surface S



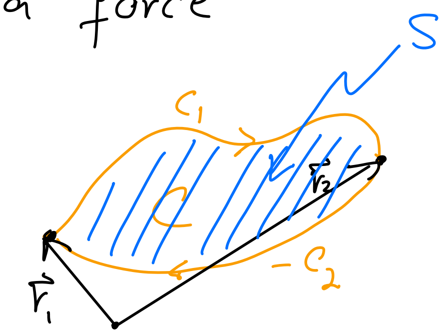
Result:

$$\oint_C \vec{F}\cdot d\vec{r} = \int_S (\vec{\nabla}\times\vec{F})_n\, da$$

"Stokes"

Definition 3:

A "conservative" force is a force satisfying



$$\oint_C W = \oint_C \vec{F}\cdot d\vec{r} = 0$$

regardless of the choice of the closed path C

Theorem 3:

Any force $\vec{F}(\vec{r})$ which can be written as $\vec{F} = \vec{\nabla} U(\vec{r}) \overset{d=2}{=} \begin{pmatrix} \frac{\partial U}{\partial x} \\ \frac{\partial U}{\partial y} \end{pmatrix}$ for a scalar function U is conservative

Proof: Choose disk D, such that $\partial D = C$

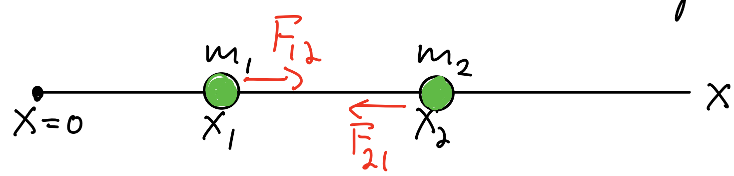
$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C \vec{\nabla} U \cdot d\vec{r} \overset{\text{Stokes' law}}{=} \int_S \vec{\nabla}\times(\vec{\nabla}U)\cdot \vec{n}\, da = 0$$

But: $\vec{\nabla}\times(\vec{\nabla}U) = \frac{\partial}{\partial x}\frac{\partial U}{\partial y} - \frac{\partial}{\partial y}\frac{\partial U}{\partial x} = 0$ $\square$

§5. Law of conservation of momentum

§5.1 Multi-particle Dynamics

Consider 2 bodies moving in one dimension:



$\longrightarrow F_e = F_{1e} + F_{2e}$

external forces

Then Newton's 2nd law gives:

1) $m_1 \ddot{x}_1 = F_{12} + F_{1e}$

force on 1 due to 2

sum of external forces on 1

2) $m_2 \ddot{x}_2 = F_{21} + F_{2e}$

force on 2 due to 1

sum of external forces on 2

$\Rightarrow$ Newton's 3rd law gives: $F_{12} = -F_{21}$

1) + 2) :

(*) $m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = \cancel{F_{12}} + F_{1e} + \cancel{F_{21}} + F_{2e} = F_e$

Rewrite (*) as:

$$M\left[\frac{m_1 \ddot{x}_1 + m_2 \ddot{x}_2}{M}\right] = F_e \quad , \quad M = m_1 + m_2$$

$$M \frac{d^2 X}{dt^2} = F_e \qquad (**)$$

where $X = \left[\frac{m_1 x_1 + m_2 x_2}{M}\right]$ weighted average

"center-of-mass coordinate CM"

In general, for N masses in 3 dimensions:

<u>Definition 1:</u> (CM)

Let N bodies with masses $m_1, \dots, m_N$ have positions $\vec{r}_1, \dots, \vec{r}_N$

→ then their center-of-mass is given by

$$\vec{R} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i}$$

Using $\vec{R}$, the generalization of (**) for N masses becomes

$$M \frac{d^2 \vec{R}}{dt^2} = \vec{F_e} \quad , \quad M = \sum_{i=1}^{N} m_i$$

internal forces have cancelled out due to Newton's 3rd law

§ 5.1 Conservation of momentum

Now consider the case $\vec{F_e} = 0$

$$\rightarrow \quad d^2 \vec{R} = 0 \quad \Rightarrow \quad \frac{d\vec{R}}{dt} = \text{const.}$$

$$\Leftrightarrow \quad M \frac{d\vec{R}}{dt} = \text{const.}$$

<u>Definition 2</u> (momentum):

The "momentum" $\vec{p}$ of a particle is defined by: $\vec{p} = m\vec{v}$

<u>Theorem 1:</u>

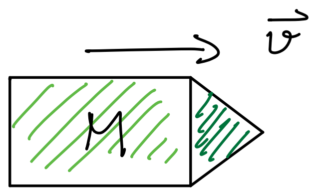
$$\xrightarrow{F_e = 0} M \frac{d^2\vec{R}}{dt^2} = \frac{d\vec{P}}{dt} = \sum_{i=1}^{N} \frac{d\vec{p}_i}{dt} = 0$$

" In the absence of external forces, the CM momentum is conserved "

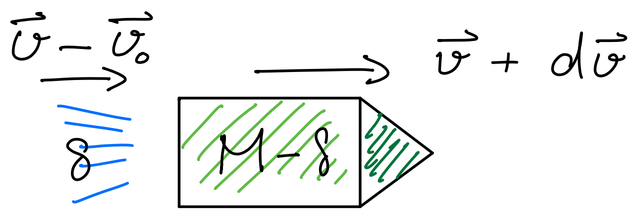
Example 1 (Rocket science):

Imagine a rocket flying with a velocity $\vec{v}$:

a)



b) the rocket increases its speeds by emitting gases at "exhaust velocity" $-v_0$ (relative to the rocket)



Let us balance the momentum before and after:

$$\cancel{M\vec{v}} = (M-\delta)(\vec{v}+d\vec{v}) + (\vec{v}-\vec{v}_0)\delta$$

$$= \cancel{M\vec{v}} + M d\vec{v} - \cancel{\vec{v}\delta} - d\vec{v}\cdot\delta + \cancel{\vec{v}\delta} - \vec{v}_0\delta$$

$$\Longleftrightarrow \quad v_0 \underset{-dM}{\underset{\|}{\delta}} = M d\vec{v} \quad \text{or} \quad -\frac{dM}{M} = \frac{dv}{v_0}$$

Integrating gives $v(t) = v_0 \log \frac{M_0}{M(t)}$ (initial rocket vel. $= 0$)